Fully supersymmetric hierarchies from an energy-dependent super Hill operator

1997 J. Phys. A: Math. Gen. 30 8661

(http://iopscience.iop.org/0305-4470/30/24/025)

J. Phys. A: Math. Gen. **30** (1997) 8661–8670. Printed in the UK PII: S0305-4470(97)87476-8

Fully supersymmetric hierarchies from an energy-dependent super Hill operator

Q P Liu†

Departamento de Física Teórica, Universidad Complutense, E28040-Madrid, Spain

Received 9 September 1997

Abstract. A super Hill operator with energy-dependent potentials is proposed and the associated integrable hierarchy is explicitly constructed explicitly. It is shown that in the general case, the resultant hierarchy is a multi-Hamiltonian system. The Miura-type transformations and modified hierarchies are also presented.

1. Introduction

During the last decade, the theory of supersymmetric (SUSY) integrable hierarchies has been an active research subject and consequently, many well known integrable equations have been extended and many supersymmetric integrable systems have been worked out. Here we cite SUSY Korteweg–de Vries (KdV) [15, 16], Kadomtsev–Petviashvili [16], Boussinesq [5] and Ablowitz–Kaup–Newell–Segur systems [21] and refer to [8, 9, 22, 23] and the references therein for more recent results.

We note that there are basically two ways to construct SUSY integrable systems: by proposing a Lax operator or by using supercomformal algebras. While in the former case, the integrability of the resultant hierarchies is guaranteed, it is not the case for the latter. Also, there are different SUSY extensions: non-extended ($N = 1$) or extended ($N \geqslant 2$). We are mainly interested in the $N = 1$ SUSY integrable systems.

In this paper, we are going to present a large number of SUSY integrable hierarchies by proposing a proper energy-dependent Hill operator. We remark that the Schrödinger equation with energy-dependent potential was first studied by Jaulent and Miodek [10] and that there in the simplest case, the associated nonlinear evolution equations were solved by means of an inverse scattering transformation. The problem is generalized in [19] and it is further shown that the resultant flows are bi-Hamiltonian systems. The remarkable multi-Hamiltonian structures have been explored and Miura-type maps are obtained in a series of papers by Antonowicz and Fordy [1–3]. The Lie algebraic reason for constructing a Miura map is provided by Marshall [18] and this subsequently leads to some new results for the Ito's system [15]. The most recent result for these hierarchies is their relationship with the zero sets of the tau function of the KdV hierarchy [17].

The generalizations of linear problems with energy potentials are interesting and begin with the third-order or Lax operator for the Boussinesq equation [4]. Unlike the Schrödinger case, one does not have arbitrary polynomial-dependent expansions here and so to have

† On leave of absence from Beijing Graduate School, China University of Mining and Technology, Beijing 100083, People's Republic of China.

interesting results, one only obtains four cases (see [4] for details). Similarly, the Toda system is generalized in this way [13].

It is interesting to generalize the idea of energy-dependent spectral operator to the super case. In this regard, Kupershmidt's spectral problem for super KdV is generalized [3]. However, Kupershmidt's KdV is not supersymmetric and it is believed that the supersymmetric systems are physically relevant.

The aim of this paper is to present SUSY integrable systems resulting from a linear super operator with energy-dependent potentials. We will show that like the Schrödinger operator case, the resultant systems are multi-Hamiltonian in nature and have multi-step modifications. Thus, the remarkable algebraic structures revealed in [1–3] are retained for our new SUSY systems. The simplest example in this construction includes one of the $N = 2$ SUSY KdV system [12].

The paper is organized as follows. In section 2, we propose the linear problem and construct the related isospectral flows. We also construct the matrix operators which are our candidates for Hamiltonian operators. In section 3, we proceed to construct the Miura-type maps which serve as a simple way to prove some of the claims made in section 2. Section 4 contains some interesting examples.

2. The linear problem

We start with the following super linear operator

$$L = \varepsilon D^3 + uD + \alpha \tag{1}$$

where $D = \partial_\vartheta + \vartheta\partial$ is the super derivation with ϑ a Grassman odd variable, $\partial = \partial/\partial x$ and $\partial_\vartheta = \partial/\partial\vartheta$; $\varepsilon = \varepsilon(\lambda)$ is a bosonic parameter depending on the spectral parameter λ; $u = u(\lambda; \vartheta, x, t)$ is the bosonic field and $\alpha = \alpha(\lambda; \vartheta, x, t)$ is fermionic field. Taking $\varepsilon = 1, u = 0$, our operator L reduces to the super Hill operator discussed in [20], so we refer to L as a super Hill operator with a slight abuse of terminology.

To obtain isospectral flows associated with L, we consider the linear problem $L\psi = 0$ together with the time evolution of the wavefunction:

$$\psi_t = P\psi \qquad P = b\partial + \beta D + c \tag{2}$$

where b and c are bosonic and β is fermionic.

By simple calculation, we have

$$\begin{aligned}
L_t - [P, L] = {} & u_t D + \alpha_t - \varepsilon(2\beta - (Db))\partial^2 + \varepsilon(b_x + (D\beta))D^3 \\
& +(\varepsilon(Db)_x - \varepsilon\beta_x + \varepsilon(Dc) + u(Db) - 2u\beta)D^2 \\
& -(bu_x + \beta(cu) - \varepsilon(D\beta)_x - \varepsilon c_x - u(D\beta))D \\
& -(b\alpha_x + \beta(D\alpha) - \varepsilon(Dc)_x - u(Dc))
\end{aligned}$$

and it is easy to see that the usual Lax equation $L_t = [P, L]$ will not lead to any consistent equation. To have meaningful results, we therefore introduce

$$Q = ((Db) - 2\beta)D + b_x + (D\beta)$$

and consider

$$\begin{aligned}
[P, L] + QL = {} & (-\varepsilon(Db)_x + \varepsilon\beta_x - \varepsilon(Dc))D^2 + (bu_x + \beta(Du) - \varepsilon(D\beta)_x \\
& -\varepsilon c_x - u(D\beta) + (Db)(Du) - 2\beta(Du) - (Db)\alpha + 2\beta\alpha \\
& +b_x u + (D\beta)u)D + b\alpha_x + \beta(D\alpha) - \varepsilon(Dc)_x - u(Dc) \\
& +(Db)(D\alpha) - 2\beta(D\alpha) + b_x\alpha + (D\beta)\alpha
\end{aligned}$$

thus, we have to choose $c = -b_x + (D\beta)$ and then

$$L_t = [P, L] + QL$$

gives us

$$\begin{aligned} u_t &= (bu)_x - \beta(Du) - 2\varepsilon(D\beta)_x + \varepsilon b_{xx} + (Db)(Du) - (Db)\alpha + 2\beta\alpha \\ \alpha_t &= (b\alpha)_x + D(\beta\alpha) + \varepsilon(Db)_{xx} - \varepsilon\beta_{xx} + u(Db)_x - u\beta_x + (Db)(D\alpha) \end{aligned}$$

which can be written neatly as

$$\mathbf{u}_t = J\mathbf{R} \tag{3}$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ \alpha \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} (Db) - \beta \\ b \end{pmatrix}$$

$$J = \begin{pmatrix} 2\varepsilon D^3 + 2\alpha - (Du) & -\varepsilon\partial^2 + \partial u - \alpha D \\ \varepsilon\partial^2 + u\partial + D\alpha & \alpha\partial + \partial\alpha \end{pmatrix}. \tag{4}$$

To obtain evolution equations, we now specify the ε, u and α in the following way

$$\varepsilon = \sum_{i=0}^{n} \varepsilon_i \lambda^i \qquad u = \sum_{i=0}^{n} u_i \lambda^i \qquad \alpha = \sum_{i=0}^{n} \alpha_i \lambda^i. \tag{5}$$

Using this choice, equation (3) can be written in the form

$$\sum_{i=0}^{n} \lambda^i \mathbf{u}_{it} = \left(\sum_{i=0}^{n} J_i \lambda^i \right) \mathbf{R} \tag{6}$$

where $\mathbf{u}_i = (u_i, \alpha_i)^{\mathrm{T}}$ and

$$J_i = \begin{pmatrix} 2\varepsilon_i D^3 + 2\alpha_i - (Du_i) & -\varepsilon_i\partial^2 + \partial u_i - \alpha_i D \\ \varepsilon_i\partial^2 + u_i\partial + D\alpha_i & \alpha_i\partial + \partial\alpha_i \end{pmatrix}. \tag{7}$$

We assume that $\mathbf{R}$ has following expansion with respect to the spectral parameter λ

$$\mathbf{R} = \sum_{i=0}^{m} \mathbf{R}_{m-i} \lambda^i$$

then the coefficients of different powers of λ of equation (6) give us

$$\begin{aligned} \mathbf{u}_{0t} &= J_0 \mathbf{R}_m \\ \mathbf{u}_{1t} &= J_0 \mathbf{R}_{m-1} + J_1 \mathbf{R}_m \\ &\vdots \\ \mathbf{u}_{nt} &= J_0 \mathbf{R}_{m-n} + J_1 \mathbf{R}_{m-n+1} + \cdots + J_n \mathbf{R}_m \end{aligned} \tag{8}$$

$$J_0 \mathbf{R}_{i-n} + J_1 \mathbf{R}_{i-n+1} + \cdots + J_n \mathbf{R}_i = 0 \qquad i = 0, \ldots, m-1. \tag{9}$$

From the above systems (8) and (9), we see that $\mathbf{R}_m$ is not determined and we have two basic cases.

(i) $u_n = -1$, $\alpha_n = 0$. In this case, the last equation of (8) takes the same form of (9) with $i = m$, which enables us to determine $\mathbf{R}_m$ in principle. This is referred to as the sKdV case.

(ii) $u_0 =$ constant, $\alpha_0 =$ (fermionic) constant. This case leads to $\mathbf{R}_m = 0$ for compatibility and is referred to as the super Harry Dym case.

Since the second case can be studied similarly, we will only consider the first case in detail.

The evolution equation (8) can be reformed as

$$\begin{pmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_{n-1} \end{pmatrix}_{t_m} = \begin{pmatrix} 0 & & J_0 \\ & \iddots & \vdots \\ J_0 & \cdots & J_{n-1} \end{pmatrix} \begin{pmatrix} \mathbf{R}_{m-n+1} \\ \vdots \\ \mathbf{R}_m \end{pmatrix} \tag{10}$$

and the recursion relation (9) can be written as

$$B_i \mathbf{R}^{(k)} = B_{i-1} \mathbf{R}^{(k+1)} \qquad i = 1, \ldots, n \tag{11}$$

where

$$B_i = \left(\begin{array}{ccc|ccc} 0 & & J_0 & & & \\ & \iddots & \vdots & & 0 & \\ J_0 & \cdots & J_{i-1} & & & \\ \hline & & & -J_{i+1} & \cdots & -J_n \\ & 0 & & \vdots & \iddots & \\ & & & -J_n & & 0 \end{array}\right) \tag{12}$$

and

$$\mathbf{R}^{(k)} = (\mathbf{R}_{k-n+1}, \ldots, \mathbf{R}_k)^{\mathrm{T}}$$

where these operators B_i are our candidates of Hamiltonian operators. In order to obtain a Hamiltonian description of the evolution system (10), we need to prove

(i) B_i are Hamiltonian operators;

(ii) $J\mathcal{R} = 0$ admits the formal power series solution $\mathcal{R} = \sum_{i=0}^{\infty} \mathbf{R}_i \lambda^{-i}$;

(iii) $\mathbf{R}^{(i)}$ can be written as variational derivatives of some functionals $\mathcal{H}_i$.

With the assumption that the above statements are proved, we now have

$$\mathbf{U}_{t_m} = B_{n-k} \delta \mathcal{H}_{m+k} \qquad k = 0, \ldots, n \tag{13}$$

where $\mathbf{U} = (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1})^{\mathrm{T}}$ and δ denotes the variational derivative with respect to $\mathbf{U}$. Thus, our system is a $(n+1)$-Hamiltonian system.

Now we can prove the statements (ii) and (iii). To this end, we introduce $\eta = D(\ln \psi)$, so that $L\psi = 0$ becomes

$$\varepsilon(\eta_x + \eta(D\eta)) + u\eta + \alpha = 0 \tag{14}$$

and it is easy to see that (14) has the solution $\eta = \sum_{-\infty}^{s} \eta_{-j} \lambda^j$ for certain values of s. It is also readily seen that each η_j provides us a conserved quantity in principle. Next, we show that the solution of (14) will supply a set solution for $J\mathcal{R} = 0$.

Proposition 1. For each solution of η of equation (14), its variational derivative, with respect to (u, α), provides us with a solution for $J\mathcal{R} = 0$.

Proof. We introduce an additional variable y so that equation (14) is written equivalently as

$$u = \varepsilon((D\eta) - y) \qquad \alpha = \varepsilon(-\eta_x - 2\eta(D\eta) + \eta y) \tag{15}$$

which serves as a map between (u, α) and (y, η). Thus, we have the following formula

$$\begin{pmatrix} \delta_y \\ \delta_\eta \end{pmatrix} = F^\dagger \begin{pmatrix} \delta_u \\ \delta_\alpha \end{pmatrix} \qquad \text{where } \delta_v = \frac{\delta}{\delta v} \tag{16}$$

and

$$F = \varepsilon \begin{pmatrix} -1 & D \\ \eta & -\partial - 2\eta D - 2(D\eta) + y \end{pmatrix}$$

is the Fréchet derivative of (15) and † denotes its adjoint.

Performing equation (16) on η and denoting $\xi = \delta_u \eta$ and $p = \delta_\alpha \eta$, we obtain

$$\xi - \eta p = 0 \qquad \varepsilon[(D\xi) + p_x + 2\eta(Dp) - 4(D\eta)p + yp] = 1. \tag{17}$$

We claim that the solution (ξ, p) of (17) provides us with a solution of $J\mathcal{R} = 0$. To prove the validity of this claim, we eliminate the variable ξ in (17) and have

$$\varepsilon[p_x + \eta(Dp) + yp - 3(D\eta)p] = 1. \tag{18}$$

Differentiating this equation leads to

$$\varepsilon[(Dp)_x + (y - 2(D\eta))(Dp) + (Dy - 3\eta_x + \eta y - 3\eta(D\eta)p] = \eta \tag{19}$$

and

$$\begin{aligned}\varepsilon[p_{xx} + (\eta_x - 2\eta y + 5\eta(D\eta))(Dp) + (y_x - 3(D\eta_x) - \eta(Dy) + 3\eta\eta_x - y^2 \\ +6y(D\eta) - 9(D\eta)^2)p] + y - 3(D\eta) = 0.\end{aligned} \tag{20}$$

Now using formulae (19) and (20) and bearing in mind the mapping (15), one can easily show that

$$2\varepsilon(D\xi_x) + 2\alpha\xi - (Du)\xi - \varepsilon p_{xx} - \alpha(Dp) + (up)_x = 0.$$

Similarly, we can check that

$$\varepsilon\xi_{xx} + u\xi_x + D(\alpha\xi) + 2\alpha p_x + \alpha_x p = 0$$

is an identity. These last two equations lead to $J\mathcal{R} = 0$ and the proposition is thus proved. □

Remark. Solvability is justified by supplying a set of solutions as above. So the strategy used here is different from the pure bosonic case, where one is able to prove this fact directly ([2]).

3. Miura maps and modifications

To construct the Miura-type map for the systems presented in the last section, we first consider the basic case: $u \to u - \lambda$ and $\alpha \to \alpha$.

By the following factorization

$$L = (D + \theta_1)(D + \theta_1 + \theta_2)(D + \theta_2)$$

we have

$$u = w_x + (D\theta) + \theta(Dw) \qquad \alpha = (Dw)_x + (D\theta)(Dw) \tag{21}$$

where we have redefined the coordinates $\theta_1 = \theta$ and $\theta_2 = Dw$ for convenience.

The Fréchet derivation of the map (21) and its adjoint are

$$m = \begin{pmatrix} \partial + \theta D & D - (Dw) \\ D\partial + (D\theta)D & (Dw)D \end{pmatrix} \qquad m^\dagger = \begin{pmatrix} -\partial + D\theta & D\partial - D(D\theta) \\ D + (Dw) & D(Dw) \end{pmatrix}$$

and we can verify that the following identity holds

$$mKm^\dagger = J$$

where $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and J is given by (4).

Modifying the map (21) with a parameter γ, we have

$$u = \gamma w_x + (D\theta) + \theta(Dw) \qquad \alpha = \gamma(Dw)_x + (D\theta)(Dw). \tag{22}$$

With these defined, we now follow the method presented in [2] and construct the Miura maps. Since the construction follows closely that presented in [2, 3], we only present the final results here. Factorizing

$$J = \sum_{i=0}^{n} \lambda^i J_i$$

in the following way

$$J = (m_0, m_1, \ldots, m_n) K \Lambda_k (m_0^\dagger, m_1^\dagger, \ldots, m_n^\dagger)^{\mathrm{T}} \tag{23}$$

where

$$m_i = \begin{pmatrix} \gamma_i \partial + \theta_i D & D - (Dw_i) \\ \gamma_i D\partial + (D\theta_i) D & (Dw_i) D \end{pmatrix} \qquad m_i^\dagger = \begin{pmatrix} -\gamma_i \partial + D\theta_i & \gamma_i D\partial - D(D\theta_i) \\ D + (Dw_i) & D(Dw_i) \end{pmatrix}$$

and

$$\Lambda_k = \left(\begin{array}{ccc|ccc} 1 & \cdots & \lambda^{k-1} & & & \\ \vdots & \iddots & & & 0 & \\ \lambda^{k-1} & & 0 & & & \\ \hline & & & 0 & & \lambda^k \\ & 0 & & & \iddots & \vdots \\ & & & \lambda^k & \cdots & \lambda^n \end{array}\right)$$

and comparing the coefficients of λ of equation (23) we have

$$\varepsilon_k = \sum_{i=0}^{k} \gamma_i \qquad k = 0, \ldots, r-1 \tag{24a}$$

$$\varepsilon_k = \sum_{i=k}^{n} \gamma_i \qquad k = r, \ldots, n \tag{24b}$$

$$u_k = \frac{1}{2} \sum_{i=0}^{k} \mathcal{W}_{i.k-i} \qquad \alpha_k = \frac{1}{2} \sum_{i=0}^{k} \Omega_{i,k-i} \qquad k = 0, \ldots, r-1 \tag{25}$$

$$u_k = \frac{1}{2} \sum_{i=0}^{n-k} \mathcal{W}_{k+i,n-i} \qquad \alpha_k = \frac{1}{2} \sum_{i=0}^{n-k} \Omega_{k+i,n-i} \qquad k = r, \ldots, n \tag{26}$$

where

$$\mathcal{W}_{i,j} = (D\theta_i) + (D\theta_j) + \gamma_i w_{j,x} + \gamma_j w_{i,x} + \theta_i (Dw_j) + \theta_j (Dw_i)$$
$$\Omega_{i,j} = (Dw_i)(D\theta_j) + (Dw_j)(D\theta_i) + \gamma_i (Dw_j)_x + \gamma_j (Dw_i)_x.$$

To reduce to the KdV case, we specify

$$u_n = -1 \qquad \alpha_n = 0$$

and we choose $\theta_n = -\vartheta$ and $w_n = 0$ for consistency. Thus, the formulae (26) become

$$u_k = -1 + (D\theta_k) + \eta_n w_{k,x} + \frac{1}{2}\sum_{i=1}^{n-k-1} \mathcal{W}_{k+i,n-i} \tag{27a}$$

$$\alpha_k = -(Dw_k) + \eta_n (Dw_k)_x + \frac{1}{2}\sum_{i=1}^{n-k-1} \Omega_{k+i,n-i}. \tag{27b}$$

Having constructed the maps, we now obtain the following proposition.

Proposition 2. Solving equations (24) for κ_i, the operators B_k (12) are related to a constant coefficient Hamiltonian operator in

$$B_k = M_k \hat{B}_k (M_k)^\dagger$$

where M_k is the Fréchet derivative of (25) and (27) and

$$\hat{B}_k = \left(\begin{array}{ccc|ccc} 0 & & -K & & & \\ & \iddots & & & 0 & \\ -K & & 0 & & & \\ \hline & & & 0 & & K \\ & 0 & & & \iddots & \\ & & & K & & 0 \end{array}\right)$$

where $\hat{B}_k$ has the same block structures as B_k (12).

Proof. Direct computation. □

Remarks.

- The Hamiltonian nature of our operators B_k is proved as a simple corollary of the above proposition for the generic case. The general case can be proved by taking the limits as in [2].
- The general Miura map (25) and (27) can be regarded as the decomposition of n step elementary maps [2]. In this way, the remarkable picture of [2] (figure 1 of [2]) reappears here.

4. Examples

In this section, we present some interesting examples. We will concentrate on the simplest cases, that is $n = 1$ and $n = 2$.

4.1. Two component case

In this case, we take $\varepsilon_0 = 1$, $\varepsilon_1 = 0$ and $u(x, t; \lambda) = u(x, t) - \lambda$ and $\alpha(x, t; \lambda) = \alpha(x, t)$. Then, we seek the formal solution $\eta = \sum_{i=1}^{\infty} \eta_i \lambda^{-i}$ of the equation

$$\eta_x + \eta(D\eta) + u\eta - \lambda\eta + \alpha = 0$$

when the first few solutions are

$$\mathcal{H}_1 = \alpha \qquad \mathcal{H}_2 = u\alpha \qquad \mathcal{H}_3 = u\alpha_x + \alpha(D\alpha + u^2)$$

which serve as the first Hamiltonians. The corresponding first non-trivial system (t_2-flow) is

$$u_t = -u_{xx} + 2uu_x + 2(D\alpha)_x \qquad \alpha_t = \alpha_{xx} + 2(u\alpha)_x. \tag{28}$$

We note that the above system reduces to the Burgers equation when $\alpha = 0$, so it can be regarded as a supersymmetric Burgers equation. The system (28) indeed is the one constructed in [6] under the name of the super two-boson system. We also comment that the next flow (t_3-flow) can be transformed to one of an $N = 2$ supersymmetric KdV equation [12] by a invertable change of coordinates [14].

The Miura map in the present case is the basic one (21) and the modified system for (28) is

$$\begin{pmatrix} w \\ \theta \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_w \\ \delta_\theta \end{pmatrix} \hat{\mathcal{H}}_2$$

where $\hat{\mathcal{H}}_2 = w_x(D\theta)(Dw) + (D\theta)^2(Dw) + (Dw)(Dw)_x\theta + (D\theta)(Dw)_x$.

As a new example, we choose ε as before and $u = u_0 + \lambda u_1$ and $\alpha = \lambda\alpha_1$ with u_0 constant. The Hamiltonian operators are

$$J_0 = \begin{pmatrix} 2D\partial & -\partial^2 + u_0\partial \\ \partial^2 + u_0\partial & 0 \end{pmatrix} \qquad J_1 = -\begin{pmatrix} 2\alpha_1 - (Du_1) & -\alpha_1 D + \partial u_1 \\ u_1\partial + D\alpha_1 & \alpha_1 D + D\alpha_1 \end{pmatrix}.$$

In this case we seek the formal solution of the form $\eta = \sum_{i=0}^{\infty} \eta_i \lambda^{-i}$ of equation (14) and the first two are

$$\mathcal{H}_0 = -\frac{\alpha_1}{u_1} \qquad \mathcal{H}_1 = u_1^{-1}\left(\left(\frac{\alpha_1}{u_1}\right)_x - \left(\frac{\alpha_1}{u_1}\right) D \left(\frac{\alpha_1}{u_1}\right) + \frac{u_0\alpha_1}{u_1} \right).$$

With $u_0 = c$(constant), we have

$$u_{1,t} = 2D\left(\frac{\alpha_1}{u_1^2}\right)_x + \left(\frac{1}{u_1}\right)_{xx} - c\left(\frac{1}{u_1}\right)_x$$
$$\alpha_{1,t} = \left(\frac{\alpha_1}{u_1^2}\right)_{xx} + c\left(\frac{\alpha_1}{u_1^2}\right)_x .$$

Interestingly, the above system admits the reduction $\alpha_1 = 0$, which means

$$v_t = -v^2(v_{xx} - cv_x) \tag{29}$$

with $v = u_1^{-1}$. Equation (29) passes the Painlevé test as shown in [7]. We also note that when $c = 0$, equation (29) is discussed in [24] and is in the list of evolution equations classified in [25] by the symmetry approach.

4.2. Four component case

Now we present the last example—the four component case: $\varepsilon = 1$, $u = u_0 + \lambda u_1 - \lambda^2$ and $\alpha = \alpha_0 + \lambda\alpha_1$. Similarly, we have the Hamiltonians

$$\mathcal{H}_1 = \alpha_1 \qquad \mathcal{H}_2 = \alpha_0 + u_1\alpha_1 \qquad \mathcal{H}_3 = u_0\alpha_1 + u_1^2\alpha_1 + u_1\alpha_0.$$

The system is tri-Hamiltonian with

$$B_0 = \begin{pmatrix} J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} J_0 & 0 \\ 0 & J_2 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}$$

where

$$J_0 = \begin{pmatrix} 2D\partial + 2\alpha_0 - (Du_0) & -\partial^2 - \alpha_0 D + \partial u_0 \\ \partial^2 + u_0\partial + D\alpha_0 & \alpha_0\partial + \partial\alpha_0 \end{pmatrix}$$
$$J_1 = \begin{pmatrix} 2\alpha_1 - (Du_1) & \alpha_1 D + \partial u_1 \\ u_1\partial + D\alpha_1 & \alpha_1\partial + \partial\alpha_1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}$$

and the first flow is

$$
\begin{aligned}
u_{0,t} &= 2(D\alpha_1)_x + 2\alpha_0\alpha_1 - (Du_0)\alpha_1 - u_{1,xx} - \alpha_0(Du_1) + (u_0u_1)_x \\
\alpha_{0,t} &= \alpha_{1,xx} + u_0\alpha_{1,x} + D(\alpha_0\alpha_1) + 2\alpha_0 u_{1,x} + \alpha_{0,x}u_1 \\
u_{1,t} &= u_{0,x} + 2u_1u_{1,x} \\
\alpha_{1,t} &= \alpha_{0,x} + 2(u_1\alpha_1)_x.
\end{aligned}
$$

The Miura map here reads as

$$
\begin{aligned}
u_0 &= w_{0,x} + (D\theta_0) + \theta_0(Dw_0) \\
\alpha_0 &= (Dw_0)_x + (D\theta_0)(Dw_0) \\
u_1 &= (D\theta_0) + (D\theta_1) - w_{0,x} + w_{1,x} + \theta_0(Dw_1) + \theta_1(Dw_0) \\
\alpha_1 &= (Dw_0)(D\theta_1) + (Dw_1)(D\theta_0) - (Dw_0)_x + (Dw_1)_x
\end{aligned}
\tag{30}
$$

and can be decomposed as follows

$$
\begin{aligned}
u_0 &= v_{0,x} + (D\mu_0) + \mu_0(Dv_0) \\
\alpha_0 &= (Dv_0)_x + (D\mu_0)(Dv_0) \\
u_1 &= v_1 \\
\alpha_1 &= \mu_1
\end{aligned}
$$

and

$$
\begin{aligned}
v_0 &= w_0 \\
\mu_0 &= \theta_0 \\
v_1 &= (D\theta_0) + (D\theta_1) - w_{0,x} + w_{1,x} + \theta_0(Dw_1) + \theta_1(Dw_0) \\
\mu_1 &= (Dw_0)(D\theta_1) + (Dw_1)(D\theta_0) - (Dw_0)_x + (Dw_1)_x.
\end{aligned}
$$

Thus, we have two step modifications here. The modified systems under all these Miura maps can be easily calculated and we will not present them here.

Remark. The Miura map (30) results from the general construction described in section 3. It is possible to derive it by linearization of the basic case. Indeed, by linearizing the basic map (21), we have

$$
\begin{aligned}
u_0 &= \hat{w}_{0,x} + (D\hat{\theta}_0) + \hat{\theta}_0(D\hat{w}_0) \\
\alpha_0 &= (D\hat{w}_0)_x + (D\hat{\theta}_0)(D\hat{w}_0) \\
u_1 &= \hat{w}_{1,x} + (D\hat{\theta}_1) + \hat{\theta}_0(D\hat{w}_1) + \hat{\theta}_1(D\hat{w}_0) \\
\alpha_1 &= (D\hat{w}_1)_x + (D\hat{\theta}_1)(D\hat{w}_0) + (D\hat{\theta}_0)(D\hat{w}_1)
\end{aligned}
$$

which is equivalent to (30) by a simple transformation, namely

$$
\hat{w}_0 = w_0 \qquad \hat{\theta}_0 = \theta_0 \qquad \hat{w}_1 = w_1 - w_0 \quad \hat{\theta}_1 = \theta_0 + \theta_1.
$$

Acknowledgments

QPL was supported by Beca para estancias temporales de doctores y tecnólogos extranjeros en España: SB95-A01722297 (Spain) and National Natural Science Foundation (China).

References

[1] Antonowicz M and Fordy A P 1987 *Physica* **28A** 345
Antonowicz M and Fordy A P 1988 *J. Phys. A: Math. Gen.* **21** L269
[2] Antonowicz M and Fordy A P 1989 *Commun. Math. Phys.* **124** 465
[3] Antonowicz M and Fordy A P 1989 *Commun. Math. Phys.* **124** 487
[4] Antonowicz M, Fordy A P and Liu Q P 1991 *Nonlinearity* **4** 669
[5] Bellucci S, Ivanov E, Krivonos S and Pichugin A 1993 *Phys. Lett.* **312B** 463
Yung C M 1993 *Phys. Lett.* **309B** 75
[6] Brunelli J C and Das A 1994 *Phys. Lett.* **337B** 303
Brunelli J C and Das A 1995 *Phys. Lett.* **354B** 307
[7] Clarkson P A, Fokas A S and Ablowitz M J 1989 *SIAM J. Appl. Math.* **49** 1188
[8] Delduc F and Gallot L 1997 $N = 2$ KP and KdV hierarchies in the extended superspace *Commun. Math. Phys.* to be published
[9] Ivanov E and Krivonos S 1997 *Phys. Lett.* **231A** 75
[10] Jaulent M and Miodek I 1976 *Lett. Math. Phys.* **1** 243
[11] Kupershmidt B A 1984 *Phys. Lett.* **102A** 213
[12] Laberge C and Mathieu P 1988 *Phys. Lett.* **215B** 718
[13] Liu Q P 1992 *J. Phys. A: Math. Gen.* **25** 3603
Liu Q P 1994 *J. Phys. A: Math. Gen.* **27** 3915
[14] Liu Q P 1997 On the integrable hierarchies associated with $N = 2$ super W_n algebra *Phys. Lett.* **A** to be published
[15] Liu Q P and Marshall I 1991 *Phys. Lett.* **160A** 155
[16] Manin Yu I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
[17] Mañas M, Martínez Alonso L and Medina E 1997 *J. Phys. A: Math. Gen.* **30** 4815
[18] Marshall I 1990 *Commun. Math. Phys.* **133** 519
Marshall I 1994 *Physica* **70D** 40
[19] Martínez Alonso L 1980 *J. Math. Phys.* **21** 2342
[20] Mathieu P 1990 *Integrable and Superintegrable Systems* ed B A Kupershmidt (Singapore: World Scientific) p 352
[21] Morosi C and Pizzocchero L 1996 *Commun. Math. Phys.* **176** 353
Morosi C and Pizzocchero L 1993 *Commun. Math. Phys.* **158** 267
[22] Popowicz Z 1996 *J. Phys. A: Math. Gen.* **29** 1281
[23] Popowicz Z 1997 Extensions of the $N = 2$ supersymmetric $a = -2$ Boussinesq hierarchy *Preprint*
[24] Rosen G 1979 *Phys. Rev.* B **19** 2398
[25] Svinolupov S I 1985 *Usp. Mat. Nauk* **40** 241
Sokolov V V, Svinolupov S I and Wolf T 1992 *Phys. Lett.* **163A** 415